SUM OF PRODUCT OF RECIPROCALS OF FIBONACCI NUMBERS

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CERTIFICATE

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This is to certify that the project report with title "SUM OF PRODUCT OF RECIPROCALS OF FIBONACCI NUMBERS" submitted by Mr. Kappagantu Prudhavi Nag, Roll No. 410MA5016, to the National Institute of Technology, Rourkela, Odisha for the partial fulfillment of the requirements of Integrated M.Sc. degree in Mathematics, is a bonafide research work carried out by him under my supervision and guidance. The content of this report in full or part has not been submitted to any other Institute or University for the award of any degree or diploma.

Gopal Krishna Panda

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ABSTRACT

Fibonacci numbers are the number sequences which follow the linear mathematical recurrence$F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2} \quad n \geq 2$. In this work, we study certain sum formulas involving products of reciprocals of Fibonacci numbers. Sum formulas with alternating signs are also studied.

TABLE OF CONTENTS

NOTATIONS

The following notations will be frequently used in this thesis.

- $\mathbb{F}_N = \sum_{n=1}^{N} \frac{1}{F_n}$
- $\mathbb{G}_N = \sum_{n=1}^{N} \frac{(-1)^n}{F_n}$
- $\mathbb{F}_N(a) = \sum_{n=1}^{N} \frac{1}{F_{n+a}}$
- $\mathbb{G}_N(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}}$
- $\mathbb{F}_N^h = \sum_{n=1}^{N} \frac{1}{F_{hn}}$
- $\mathbb{G}_N^h = \sum_{n=1}^{N} \frac{(-1)^n}{F_{hn}}$
- $\mathbb{H}_N = \sum_{n=1}^{N} \frac{1}{F_{-n}}$
- $\mathbb{I}_N = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n}}$
- $\mathbb{H}_N(a) = \sum_{n=1}^{N} \frac{1}{F_{-n-a}}$
- $\mathbb{I}_N(a) = \sum_{n=1}^{N} \frac{1}{F_{-n-a}}$
- $\mathbb{H}_N(0,a) = \sum_{n=1}^{N} \frac{1}{F_{-n}F_{-n-a}}$
- $\mathbb{I}_N(0,a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n}F_{-n-a}}$
- $\mathbb{H}_N(a,b) = \sum_{n=1}^{N} \frac{1}{F_{-n-a}F_{-n-b}}$
- $\mathbb{I}_N(a,b) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n-a}F_{-n-b}}$
- $\mathbb{H}_N^h = \sum_{n=1}^{N} \frac{1}{F_{-hn}}$
- $\mathbb{I}_N^h = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-hn}}$
- $\mathbb{H}_N^h(a) = \sum_{n=1}^{N} \frac{1}{F_{-hn-a}}$
- $\mathbb{I}_N^h(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-hn-a}}$

CHAPTER 1

INTRODUCTION

Fibonacci sequence of numbers is one of the most intriguing number sequence in mathematics. The series is named after the famous Italian Mathematician Fibonacci of the Bonacci family. He is also known as the Leonardo of Pisa. The following problem proposed by Fibonacci himself gave birth to the sequence.

The Fibonacci Problem: Suppose there are two newborn rabbits, one male and one female. Find the number of rabbits produced in a year if [1]

- Each pair takes one month to become mature
- Each pair produces a mixed pair every month from the second month
- All rabbits are immortal

Solution: For convenience let us assume that the rabbits are born on January 1st and we need to find the number of rabbits on December 1st. The table below is used to find the solution of the problem.

No. of Pairs	Jan	Feb	Mar	April	May	June	July	Aug	Sep	Oct	Nov	Dec
Adults	0	1	1	2	3	5	8	13	21	34	55	89
Babies	1	0	1	1	2	3	5	8	13	21	34	34
Total	1	1	2	3	5	8	13	21	34	55	89	144

From the above table it is evident that the number of rabbits at the end of the year are 144. If observed closely it is observed that the new number is equal to the sum of the previous two numbers.

MATHEMATICS OF FIBONACCI NUMBERS

The numbers in the bottom row are called the Fibonacci numbers. From the table a recursive relation is yielded as below

$$F_n = F_{n-1} + F_{n-2}\,, \qquad n \geq 2.$$

where $F_0 = 0$ and $F_1 = 1$ [2]. Sometimes it is customary to start the Fibonacci numbers from F_1 instead of F_0. Then the initial two conditions become $F_1 = 1$ and $F_2 = 1$. With any of the above two conditions the series generated is the same.

It is surprising that Fibonacci numbers can be extracted from Pascal's triangle. The above observation was confirmed by Lucas in 1876 when he derived a straightforward formula to find the Fibonacci numbers [1]

$$F_n = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-i-1}{i}, \quad n \geq 1.$$

In 1843 a French mathematician named Jacques Philippe Marie Binet invented a way to calculate the n^{th} Fibonacci numbers. If $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ then $F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$ [3].

The above formula shows an interesting aspect that the Fibonacci number can be written in terms of the golden ratio. Fibonacci numbers appear in many places in both nature and mathematics. They occur in music, geography, nature, and geometry. They can be found in the spiral arrangements of seeds of sunflowers, the scale patterns of pine cones, the arrangement of leaves and the number of petals on the flower.

FIBONACCI NUMBERS WITH NEGATIVE INDICES

The negative subscript of the Fibonacci numbers can be converted to positive subscript as indicated in [4]. The recurrence relation for the negative Fibonacci numbers is as follows:

$$F_{-n} = F_{2-n} - F_{1-n}, \quad n \geq 2$$

The initial conditions for these numbers are $F_0 = 0$ and $F_1 = 1$. It is observed that the negative Fibonacci numbers have the same initial conditions as of the positive Fibonacci numbers.

Rabinowitz [5] stated that the alternating general summation of order 1 is given by

$$\mathbb{G}_N(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}} = \mathbb{G}_{N+a} - \mathbb{G}_a.$$

We disprove the above identity using a counter example.

Now let us take $a = 3$ and $N = 5$ and find the value of $\mathbb{G}_N(a)$.

$$\sum_{n=1}^{5} \frac{(-1)^n}{F_{n+3}} = \frac{-1}{F_4} + \frac{1}{F_5} + \frac{-1}{F_6} + \frac{1}{F_7} + \frac{-1}{F_8}$$

$$\mathbb{G}_8 = \frac{-1}{F_1} + \frac{1}{F_2} + \frac{-1}{F_3} + \frac{1}{F_4} + \frac{-1}{F_5} + \frac{1}{F_6} + \frac{-1}{F_7} + \frac{1}{F_8}$$

$$\mathbb{G}_3 = \frac{-1}{F_1} + \frac{1}{F_2} + \frac{-1}{F_3}$$

$$\mathbb{G}_{N+a} - \mathbb{G}_a = \frac{1}{F_4} + \frac{-1}{F_5} + \frac{1}{F_6} + \frac{-1}{F_7} + \frac{1}{F_8} \neq \mathbb{G}_N(a).$$

It can be checked that the identity for $\mathbb{G}_N(a)$ is wrong when a is odd and correct when a is even.

Claim: If $a > 0$ then $\mathbb{G}_N(a) = \begin{cases} \mathbb{G}_a - \mathbb{G}_{N+a} & \text{if } a \text{ is odd,} \\ \mathbb{G}_{N+a} - \mathbb{G}_a & \text{if } a \text{ is even.} \end{cases}$

Proof:

$$\mathbb{G}_{N+a} - \mathbb{G}_a = \sum_{n=1}^{N+a} \frac{(-1)^n}{F_n} - \sum_{n=1}^{a} \frac{(-1)^n}{F_n} = \sum_{n=a+1}^{N+a} \frac{(-1)^n}{F_n}.$$

Case 1. a is odd, N is odd

$$\mathbb{G}_N(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}} = \frac{-1}{F_{1+a}} + \frac{1}{F_{2+a}} + \cdots + \frac{1}{F_{(N-1)+a}} + \frac{-1}{F_{N+a}}.$$

$$\begin{aligned} \mathbb{G}_{N+a} - \mathbb{G}_a &= \sum_{n=a+1}^{N+a} \frac{(-1)^n}{F_n} \\ &= \frac{1}{F_{1+b}} + \frac{-1}{F_{2+a}} + \cdots + \frac{-1}{F_{(N-1)+a}} + \frac{1}{F_{N+a}} \\ &= -\left(\frac{-1}{F_{1+a}} + \frac{1}{F_{2+a}} + \cdots + \frac{1}{F_{(N-1)+a}} + \frac{-1}{F_{N+a}}\right) \\ &= -\mathbb{G}_N(a). \end{aligned}$$

Case 2. a is odd, N is even

$$\mathbb{G}_N(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}} = \frac{-1}{F_{1+a}} + \frac{1}{F_{2+a}} + \cdots + \frac{-1}{F_{(N-1)+a}} + \frac{1}{F_{N+a}}.$$

$$\begin{aligned} \mathbb{G}_{N+a} - \mathbb{G}_b &= \sum_{n=a+1}^{N+a} \frac{(-1)^n}{F_n} \\ &= \frac{1}{F_{1+a}} + \frac{-1}{F_{2+a}} + \cdots + \frac{1}{F_{(N-1)+a}} + \frac{-1}{F_{N+a}} \\ &= -\left(\frac{-1}{F_{1+a}} + \frac{1}{F_{2+a}} + \cdots + \frac{-1}{F_{(N-1)+a}} + \frac{1}{F_{N+a}}\right) \\ &= -\mathbb{G}_N(a). \end{aligned}$$

Case 3. a is even , N is odd

$$\mathbb{G}_N(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}} = \frac{-1}{F_{1+a}} + \frac{1}{F_{2+a}} + \cdots + \frac{1}{F_{(N-1)+a}} + \frac{-1}{F_{N+a}}.$$

$$\begin{aligned}\mathbb{G}_{N+a} - \mathbb{G}_b &= \sum_{n=a+1}^{N+a} \frac{(-1)^n}{F_n} \\ &= \frac{-1}{F_{1+a}} + \frac{1}{F_{2+a}} + \cdots + \frac{1}{F_{(N-1)+a}} + \frac{-1}{F_{N+a}} \\ &= \mathbb{G}_N\,(a).\end{aligned}$$

Case 4. a is even , N is even

$$\mathbb{G}_N(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}} = \frac{-1}{F_{1+a}} + \frac{1}{F_{2+a}} + \cdots + \frac{-1}{F_{(N-1)+a}} + \frac{1}{F_{N+a}}.$$

$$\begin{aligned}\mathbb{G}_{N+a} - \mathbb{G}_b &= \sum_{n=a+1}^{N+a} \frac{(-1)^n}{F_n} \\ &= \frac{-1}{F_{1+a}} + \frac{1}{F_{2+a}} + \cdots + \frac{-1}{F_{(N-1)+a}} + \frac{1}{F_{N+a}} \\ &= \mathbb{G}_N\,(a).\end{aligned}$$

$\blacksquare$

CHAPTER 2

SUM OF RECIPROCALS OF FIBONACCI NUMBERS WITH POSITIVE INDICES

The following notations for the alternating and non-alternating sum of k^{th} order are available in [5].

$$S(a_1, a_2, \dots, a_{k-1}, a_k) = \sum_{n=1}^{N} \frac{1}{F_{n+a_1} F_{n+a_2 \dots F_{n+a_k}}}$$

$$T(a_1, a_2, \dots, a_{k-1}, a_k) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a_1} F_{n+a_2 \dots F_{n+a_k}}}$$

The sum S is called the non-alternating summation of order k. The second sum T is called the alternating sum of order k. In both the cases $0 < a_1 < a_2 < \cdots < a_{k-1} < a_k$.

ORDER 2

We consider first the problem of finding the following second order sums.

$$\mathbb{F}_N(a, b) = \sum_{n=1}^{N} \frac{1}{F_{n+a} F_{n+b}}$$

$$\mathbb{G}_N(a, b) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a} F_{n+b}}$$

NON-ALTERNATING SUM

For $a > 0$, Rabinowitz [5] got the following formula.

$$\mathbb{F}_N(0, a) = \sum_{n=1}^{N} \frac{1}{F_n F_{n+a}} = \begin{cases} \dfrac{1}{F_a} \displaystyle\sum_{i=1}^{\lfloor a/2 \rfloor} \left(\frac{1}{F_{N+2i} F_{N+2i+1}} - \frac{1}{F_{2i} F_{2i+1}} \right) + \frac{\mathbb{K}_N}{F_a} & \text{if } a \text{ is odd,} \\ \dfrac{1}{F_a} \displaystyle\sum_{i=1}^{a/2} \left(\frac{1}{F_{2i-1} F_{2i}} - \frac{1}{F_{N+2i-1} F_{N+2i}} \right) & \text{if } a \text{ is even.} \end{cases}$$

where $\mathbb{K}_N = \sum_{i=1}^{N} \frac{1}{F_n F_{n+1}}$. The above sum is called the non-alternating sum of order 2. The aim is to find an equivalent expression of $\mathbb{F}_N(a, b)$. To do this we use the help of the following result.

Theorem 1: *For* $0 < a < b$ *then* $F_N\,(a, b) = F_{N+a}\,(0, b-a) - F_a\,(0, b-a)$.

Proof: We start from the right hand side of the equation and come to the left hand side.

$$
\begin{aligned}
\mathbb{F}_{N+a}(0, b-a) - \mathbb{F}_a(0, b-a) &= \sum_{n=1}^{N+a} \frac{1}{F_n F_{n+b-a}} - \sum_{i=1}^{a} \frac{1}{F_n F_{n+b-a}} \\
&= \sum_{n=a+1}^{N+a} \frac{1}{F_n F_{n+b-a}} \\
&= \frac{1}{F_{a+1} F_{b+1}} + \frac{1}{F_{a+2} F_{b+2}} + \cdots + \frac{1}{F_{N+a-1} F_{N+b-1}} + \frac{1}{F_{N+a} F_{N+b}} \\
&= \sum_{n=1}^{N} \frac{1}{F_{n+a} F_{n+b}} \\
&= \mathbb{F}_N(a, b)
\end{aligned}
$$

■

By using the above theorem and the formula for $\mathbb{F}_N(0, a)$ stated by Rabinowitz [5] the expression for $\mathbb{F}_N(a, b)$ is calculated.

Theorem 2: *If* $0 < a < b$ *then*

$$
\mathbb{F}_N(a, b) = \begin{cases} \dfrac{1}{F_{b-a}} \displaystyle\sum_{i=1}^{\lfloor (b-a)/2 \rfloor} \left(\frac{1}{F_{N+a+2i} F_{N+a+2i+1}} - \frac{1}{F_{a+2i} F_{a+2i+1}} \right) + \frac{\mathbb{F}_N(a, a+1)}{F_{b-a}} & \text{if } (b-a) \text{is odd,} \\ \dfrac{1}{F_{b-a}} \displaystyle\sum_{i=1}^{(b-a)/2} \left(\frac{1}{F_{a+2i-1} F_{a+2i}} - \frac{1}{F_{N+a+2i-1} F_{N+a+2i}} \right) & \text{if } (b-a) \text{is even.} \end{cases}
$$

Proof: We take the help of Theorem 1 to prove this theorem.

$$
\mathbb{F}_{N+a}(0, b-a) = \begin{cases} \dfrac{1}{F_{b-a}} \displaystyle\sum_{i=1}^{\lfloor (b-a)/2 \rfloor} \left(\frac{1}{F_{N+a+2i} F_{N+a+2i+1}} - \frac{1}{F_{2i} F_{2i+1}} \right) + \frac{\mathbb{K}_{N+a}}{F_{b-a}} & \text{if } (b-a) \text{ is odd,} \\ \dfrac{1}{F_{b-a}} \displaystyle\sum_{i=1}^{b(b-a)/2-a/2} \left(\frac{1}{F_{2i-1} F_{2i}} - \frac{1}{F_{N+a+2i-1} F_{N+a+2i}} \right) & \text{if } (b-a) \text{ is even.} \end{cases}
$$

$$
\mathbb{F}_a(0, b-a) = \begin{cases} \dfrac{1}{F_{b-a}} \displaystyle\sum_{i=1}^{\lfloor (b-a)/2 \rfloor} \left(\frac{1}{F_{a+2i} F_{a+2i+1}} - \frac{1}{F_{2i} F_{2i+1}} \right) + \frac{\mathbb{K}_a}{F_{b-a}} & \text{if } (b-a) \text{ is odd,} \\ \dfrac{1}{F_{b-a}} \displaystyle\sum_{i=1}^{(b-a)/2} \left(\frac{1}{F_{2i-1} F_{2i}} - \frac{1}{F_{a+2i-1} F_{a+2i}} \right) & \text{if } (b-a) \text{ is even.} \end{cases}
$$

Depending upon the parity of $(b-a)$, two different cases are taken into consideration

Case 1. $\boldsymbol{(b-a)}$ **is odd**

In this case,

$$\mathbb{F}_N(a,b)=\frac{1}{F_{b-a}}\sum_{i=1}^{\lfloor (b-a)/2\rfloor}\left(\frac{1}{F_{N+a+2i}F_{N+a+2i+1}}-\frac{1}{F_{2i}F_{2i+1}}-\frac{1}{F_{a+2i}F_{a+2i+1}}+\frac{1}{F_{2i}F_{2i+1}}\right)+\frac{\mathbb{K}_{N+a}-\mathbb{K}_a}{F_{b-a}}$$
$$=\frac{1}{F_{b-a}}\sum_{i=1}^{\lfloor (b-a)/2\rfloor}\left(\frac{1}{F_{N+a+2i}F_{N+a+2i+1}}-\frac{1}{F_{a+2i}F_{a+2i+1}}\right)+\frac{\mathbb{K}_{N+a}-\mathbb{K}_a}{F_{b-a}}.$$

Now,

$$\mathbb{K}_{N+a}-\mathbb{K}_N=\sum_{n=1}^{N+a}\frac{1}{F_nF_{n+1}}-\sum_{n=1}^{a}\frac{1}{F_nF_{n+1}}=\sum_{n=a+1}^{N+a}\frac{1}{F_nF_{n+1}}=\sum_{n=1}^{N}\frac{1}{F_{n+a}F_{n+a+1}}=\mathbb{F}_N(a,a+1)$$
$$\mathbb{F}_N(a,b)=\frac{1}{F_{b-a}}\sum_{i=1}^{\lfloor (b-a)/2\rfloor}\left(\frac{1}{F_{N+a+2i}F_{N+a+2i+1}}-\frac{1}{F_{a+2i}F_{a+2i+1}}\right)+\frac{\mathbb{F}_N(a,a+1)}{F_{b-a}}.$$

Case 2. $(b-a)$ is even

$$\mathbb{F}_N(a,b)=\frac{1}{F_{b-a}}\sum_{i=1}^{(b-a)/2}\left(\frac{1}{F_{2i-1}F_{2i}}-\frac{1}{F_{N+a+2i-1}F_{N+a+2i}}-\frac{1}{F_{2i-1}F_{2i}}-\frac{1}{F_{a+2i-1}F_{a+2i}}\right)$$
$$=\frac{1}{F_{b-a}}\sum_{i=1}^{(b-a)/2}\left(\frac{1}{F_{a+2i-1}F_{a+2i}}-\frac{1}{F_{N+a+2i-1}F_{N+a+2i}}\right).$$

■

ALTERNATING SUM

Let $a>0$. The following identity is available in from [6].

$$\mathbb{G}_N(0,a)=\sum_{n=1}^{N}\frac{(-1)^n}{F_nF_{n+a}}=\frac{1}{F_a}\sum_{j=1}^{a}\left(\frac{F_{j-1}}{F_j}-\frac{F_{N+j-1}}{F_{N+j}}\right).$$

The above is a sum with alternating sign (called an alternating sum) of order 2. We use the above sum to find a formula for $\mathbb{G}_N(a,b)$. To achieve this, we use of the following result.

Theorem 3: *If* $0<a<b$ *then*

$$\mathbb{G}_N(a,b)=\begin{cases}\mathbb{G}_a(0,b-a)-\mathbb{G}_{N+a}(0,b-a) & \text{if } a \text{ is odd},\\ \mathbb{G}_{N+a}(0,b-a)-\mathbb{G}_a(0,b-a) & \text{if } a \text{ is even}.\end{cases}$$

Proof: Observe that

$$\mathbb{G}_{N+a}(0,b-a)-\mathbb{G}_a(0,b-a)=\sum_{n=1}^{N+a}\frac{(-1)^n}{F_NF_{n+b-a}}-\sum_{n=1}^{a}\frac{(-1)^n}{F_NF_{n+b-a}}=\sum_{n=a+1}^{N+a}\frac{(-1)^n}{F_nF_{n+b-a}}.$$

We distinguish four cases:

Case 1. a and N are odd

$$\mathbb{G}_N(a,b)=\sum_{n=1}^{N}\frac{(-1)^n}{F_{n+a}F_{n+b}}=\frac{-1}{F_{a+1}F_{b+1}}+\frac{1}{F_{a+2}F_{b+2}}+\cdots+\frac{1}{F_{N+a-1}F_{N+b-1}}+\frac{-1}{F_{N+a}F_{N+b}}.$$

$$\begin{aligned}\mathbb{G}_{N+a}(0,b-a)-\mathbb{G}_a(0,b-a)&=\sum_{n=a+1}^{N+a}\frac{(-1)^n}{F_nF_{n+b-a}}\\&=\frac{1}{F_{a+1}F_{b+2}}+\frac{-1}{F_{a+2}F_{b+3}}+\cdots+\frac{-1}{F_{N+a-1}F_{N+b-1}}+\frac{1}{F_{N+a}F_{N+b}}\\&=-\left(\frac{-1}{F_{a+1}F_{b+1}}+\frac{1}{F_{a+2}F_{b+2}}+\cdots+\frac{1}{F_{N+a-1}F_{N+b-1}}+\frac{-1}{F_{N+a}F_{N+b}}\right)\\&=-\mathbb{G}_N(a,b).\end{aligned}$$

Case 2. a is odd and N is even

$$\mathbb{G}_N(a,b)=\sum_{n=1}^{N}\frac{(-1)^n}{F_{n+a}F_{n+b}}=\frac{-1}{F_{a+1}F_{b+1}}+\frac{1}{F_{a+2}F_{b+2}}+\cdots+\frac{-1}{F_{N+a-1}F_{N+b-1}}+\frac{1}{F_{N+a}F_{N+b}}.$$

$$\begin{aligned}\mathbb{G}_{N+a}(0,b-a)-\mathbb{G}_a(0,b-a)&=\sum_{n=a+1}^{N+a}\frac{(-1)^n}{F_nF_{n+b-a}}\\&=\frac{1}{F_{a+1}F_{b+2}}+\frac{-1}{F_{a+2}F_{b+3}}+\cdots+\frac{1}{F_{N+a-1}F_{N+b-1}}+\frac{-1}{F_{N+a}F_{N+b}}\\&=-\left(\frac{-1}{F_{a+1}F_{b+1}}+\frac{1}{F_{a+2}F_{b+2}}+\cdots+\frac{-1}{F_{N+a-1}F_{N+b-1}}++\frac{1}{F_{N+a}F_{N+b}}\right)\\&=-\mathbb{G}_N(a,b).\end{aligned}$$

Case 3. a is even and N is odd

$$\mathbb{G}_N(a,b)=\sum_{n=1}^{N}\frac{(-1)^n}{F_{n+a}F_{n+b}}=\frac{-1}{F_{a+1}F_{b+1}}+\frac{1}{F_{a+2}F_{b+2}}+\cdots+\frac{1}{F_{N+a-1}F_{N+b-1}}+\frac{-1}{F_{N+a}F_{N+b}}.$$

$$\begin{aligned}\mathbb{G}_{N+a}(0,b-a)-\mathbb{G}_a(0,b-a)&=\sum_{n=a+1}^{N+a}\frac{(-1)^n}{F_nF_{n+b-a}}\\&=\frac{-1}{F_{a+1}F_{b+1}}+\frac{1}{F_{a+2}F_{b+2}}+\cdots+\frac{1}{F_{N+a-1}F_{N+b-1}}++\frac{-1}{F_{N+a}F_{N+b}}\end{aligned}$$

$$= \mathbb{G}_N(a,b).$$

Case 4. a and N are even

$$\mathbb{G}_N(a,b) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{n+a}F_{n+b}} = \frac{-1}{F_{a+1}F_{b+1}} + \frac{1}{F_{a+2}F_{b+2}} + \cdots + \frac{-1}{F_{N+a-1}F_{N+b-1}} + \frac{1}{F_{N+a}F_{N+b}}$$

$$\begin{aligned}\mathbb{G}_{N+a}(0,b-a) - \mathbb{G}_a(0,b-a) &= \sum_{n=a+1}^{N+a} \frac{(-1)^n}{F_nF_{n+b-a}} \\ &= \frac{-1}{F_{a+1}F_{b+1}} + \frac{1}{F_{a+2}F_{b+2}} + \cdots + \frac{-1}{F_{N+a-1}F_{N+b-1}} + + \frac{1}{F_{N+a}F_{N+b}} \\ &= \mathbb{G}_N(a,b).\end{aligned}$$

■

The following theorem is one of the main results of this chapter.

Theorem 4: *If* $0 < a < b$ *then*

$$\mathbb{G}_N(a,b) = \begin{cases} \dfrac{1}{F_{b-a}} \displaystyle\sum_{j=1}^{b-a} \left(\frac{F_{N+a+j-1}}{F_{N+a+j}} - \frac{F_{a+j-1}}{F_{a+j}} \right) & \text{if } a \text{ is odd,} \\ \dfrac{1}{F_{b-a}} \displaystyle\sum_{j=1}^{b-a} \left(\frac{F_{a+j-1}}{F_{a+j}} - \frac{F_{N+a+j-1}}{F_{N+a+j}} \right) & \text{if } a \text{ is even.} \end{cases}$$

Proof: We separate two cases:

Case 1. a is odd

$$\begin{aligned}\mathbb{G}_a(0,b-a) - \mathbb{G}_{N+a}(0,b-a) &= \frac{1}{F_{b-a}} \sum_{j=1}^{b-a} \left(\frac{F_{j-1}}{F_j} - \frac{F_{a+j-1}}{F_{a+j}} \right) - \frac{1}{F_{b-a}} \sum_{j=1}^{b-a} \left(\frac{F_{j-1}}{F_j} - \frac{F_{N+a+j-1}}{F_{N+a+j}} \right) \\ &= \frac{1}{F_{b-a}} \sum_{j=1}^{b-a} \left(\frac{F_{N+a+j-1}}{F_{N+a+j}} - \frac{F_{a+j-1}}{F_{a+j}} \right).\end{aligned}$$

Case 2. a is even

$$\begin{aligned}\mathbb{G}_{N+a}(0,b-a) - \mathbb{G}_a(0,b-a) &= \frac{1}{F_{b-a}} \sum_{j=1}^{b-a} \left(\frac{F_{j-1}}{F_j} - \frac{F_{N+a+j-1}}{F_{N+a+j}} \right) - \frac{1}{F_{b-a}} \sum_{j=1}^{b-a} \left(\frac{F_{j-1}}{F_j} - \frac{F_{a+j-1}}{F_{a+j}} \right) \\ &= \frac{1}{F_{b-a}} \sum_{j=1}^{b-a} \left(\frac{F_{a+j-1}}{F_{a+j}} - \frac{F_{N+a+j-1}}{F_{N+a+j}} \right).\end{aligned}$$

■

SUM WITH INDICES IN A.P.

Let h be a natural number. We express certain sums in terms of the following two sums.

$$\mathbb{F}_N^h = \sum_{n=1}^{N} \frac{1}{F_{hn}}, \qquad \mathbb{G}_N^h = \sum_{n=1}^{N} \frac{(-1)^n}{F_{hn}}$$

NON-ALTERNATING SUM OF ORDER 1

We consider the problem of finding $\mathbb{F}_N^h(a)$ in terms of $\mathbb{F}_N^h$.

$Theorem$ 5: $If\ a > 0\ and\ h|a\ then\ \mathbb{F}_N^h(a) = \mathbb{F}_{N+\frac{a}{h}}^h - \mathbb{F}_{\frac{a}{h}}^h$.

$Proof$: Let $b = \frac{a}{h}$ and $a = bh$ then,

$$\begin{aligned}
\mathbb{F}_{N+\frac{a}{h}}^h - \mathbb{F}_{\frac{a}{h}}^h &= \sum_{n=1}^{N+b} \frac{1}{F_{hn}} - \sum_{n=1}^{b} \frac{1}{F_{hn}} = \sum_{n=b+1}^{N+b} \frac{1}{F_{hn}} \\
&= \frac{1}{F_{h(b+1)}} + \frac{1}{F_{h(b+2)}} + \cdots + \frac{1}{F_{h(N+b-1)}} + \frac{1}{F_{h(N+b)}} \\
&= \frac{1}{F_{h+a}} + \frac{1}{F_{2h+a}} + \cdots + \frac{1}{F_{(N-1)h+a}} + \frac{1}{F_{Nh+a}} \\
&= \mathbb{F}_N^h(a).
\end{aligned}$$

■

ALTERNATING SUM OF ORDER 1

We consider the problem of finding $\mathbb{G}_N^h(a)$ in terms of $\mathbb{G}_N^h$.

$Theorem$ 6: $If\ a > 0\ and\ h|a\ then$

$$\mathbb{G}_N^h(a) = \begin{cases} \mathbb{G}_{\frac{a}{h}}^h - \mathbb{G}_{N+\frac{a}{h}}^h & \text{if } a \text{ is odd,} \\ \mathbb{G}_{N+\frac{a}{h}}^h - \mathbb{G}_{\frac{a}{h}}^h & \text{if } a \text{ is even.} \end{cases}$$

Proof: Let $b = \frac{a}{h}$ and $a = bh$ then,

$$\mathbb{G}_{N+b}^h - \mathbb{G}_b^h = \sum_{n=1}^{N+b} \frac{(-1)^n}{F_{hn}} - \sum_{n=1}^{b} \frac{(-1)^n}{F_{hn}} = \sum_{n=b+1}^{N+b} \frac{(-1)^n}{F_{hn}}.$$

Once again we distinguish four cases.

Case 1. $b = \frac{a}{h}$ and N are odd.

$$\mathbb{G}_N^h(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{hn+a}} = \frac{-1}{F_{h+a}} + \frac{1}{F_{2h+a}} + \cdots + \frac{1}{F_{(N-1)h+a}} + \frac{-1}{F_{Nh+a}}.$$

$$\begin{aligned}
\mathbb{G}_{N+b}^h - \mathbb{G}_b^h &= \sum_{n=b+1}^{N+b} \frac{(-1)^n}{F_{hn}} \\
&= \frac{1}{F_{h+hb}} + \frac{-1}{F_{2h+hb}} + \cdots + \frac{-1}{F_{(N-1)h+hb}} + \frac{1}{F_{Nh+hb}} \\
&= \frac{1}{F_{h+a}} + \frac{-1}{F_{2h+a}} + \cdots + \frac{-1}{F_{(N-1)h+a}} + \frac{-1}{F_{Nh+a}} \\
&= -\left(\frac{-1}{F_{h+a}} + \frac{1}{F_{2h+a}} + \cdots + \frac{1}{F_{(N-1)h+a}} + \frac{-1}{F_{Nh+a}}\right) \\
&= -\mathbb{G}_N^h(a).
\end{aligned}$$

Case 2. $b = \frac{a}{h}$ is odd, N is even

$$\mathbb{G}_N^h(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{hn+a}} = \frac{-1}{F_{h+a}} + \frac{1}{F_{2h+a}} + \cdots + \frac{-1}{F_{(N-1)h+a}} + \frac{1}{F_{Nh+a}}.$$

$$\begin{aligned}
\mathbb{G}_{N+b}^h - \mathbb{G}_b^h &= \sum_{n=b+1}^{N+b} \frac{(-1)^n}{F_{hn}} \\
&= \frac{1}{F_{h+hb}} + \frac{-1}{F_{2h+hb}} + \cdots + \frac{1}{F_{(N-1)h+hb}} + \frac{-1}{F_{Nh+hb}} \\
&= \frac{1}{F_{h+a}} + \frac{-1}{F_{2h+a}} + \cdots + \frac{1}{F_{(N-1)h+a}} + \frac{-1}{F_{Nh+a}} \\
&= -\left(\frac{-1}{F_{h+a}} + \frac{1}{F_{2h+a}} + \cdots + \frac{1}{F_{(N-1)h+a}} + \frac{-1}{F_{Nh+a}}\right) \\
&= -\mathbb{G}_N^h(a)
\end{aligned}$$

Case 3. $b = \frac{a}{h}$ is even , N is odd

$$\mathbb{G}_N^h(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{hn+a}} = \frac{-1}{F_{h+a}} + \frac{1}{F_{2h+a}} + \cdots + \frac{1}{F_{(N-1)h+a}} + \frac{-1}{F_{Nh+a}}.$$

$$\begin{aligned}
\mathbb{G}_{N+b}^h - \mathbb{G}_b^h &= \sum_{n=b+1}^{N+b} \frac{(-1)^n}{F_{hn}} \\
&= \frac{-1}{F_{h+hb}} + \frac{1}{F_{2h+hb}} + \cdots + \frac{1}{F_{(N-1)h+hb}} + \frac{-1}{F_{Nh+hb}} \\
&= \frac{1}{F_{h+a}} + \frac{-1}{F_{2h+a}} + \cdots + \frac{1}{F_{(N-1)h+a}} + \frac{-1}{F_{Nh+a}} \\
&= \mathbb{G}_N^h(a).
\end{aligned}$$

Case 4. $b = \frac{a}{h}$ and N are even

$$\mathbb{G}_N^h(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{hn+a}} = \frac{-1}{F_{h+a}} + \frac{1}{F_{2h+a}} + \cdots + \frac{-1}{F_{(N-1)h+a}} + \frac{1}{F_{Nh+a}}.$$

$$\begin{aligned} \mathbb{G}_{N+b}^h - \mathbb{G}_b^h &= \sum_{n=b+1}^{N+b} \frac{(-1)^n}{F_{hn}} \\ &= \frac{-1}{F_{h+hb}} + \frac{1}{F_{2h+hb}} + \cdots + \frac{-1}{F_{(N-1)h+hb}} + \frac{1}{F_{Nh+hb}} \\ &= \frac{1}{F_{h+a}} + \frac{-1}{F_{2h+a}} + \cdots + \frac{-1}{F_{(N-1)h+a}} + \frac{1}{F_{Nh+a}} \\ &= \mathbb{G}_N^h(a). \end{aligned}$$

■

CHAPTER 3

SUM OF RECIPROCALS OF FIBONACCI NUMBERS WITH NEGATIVE INDICES

We use the conversion of negative Fibonacci numbers with negative indices to Fibonacci numbers with positive indices and then derive the identities for these numbers. We use the following notations.

$$\mathbb{H}_N = \sum_{n=1}^{N} \frac{1}{F_{-n}} \text{ , } \mathbb{I}_N = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n}}$$

We first write them in terms of $\mathbb{F}_N$ and $\mathbb{G}_N$. We use the formula

$$F_{-n} = (-1)^{n+1} F_n$$

stated in [3]. Then we have

$$\mathbb{H}_N = \sum_{n=1}^{N} \frac{1}{F_{-n}} = \sum_{n=1}^{N} \frac{1}{(-1)^{n+1} F_n} = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{F_n} = -\sum_{n=1}^{N} \frac{(-1)^n}{F_n} = -\mathbb{G}_N,$$

$$\mathbb{I}_N = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n}} = \sum_{n=1}^{N} \frac{(-1)^n}{(-1)^{n+1} F_n} = \sum_{n=1}^{N} \frac{(-1)^1}{F_n} = -\sum_{n=1}^{N} \frac{1}{F_n} = -\mathbb{F}_N.$$

■

ORDER 1

We first consider the following sums:

$$\mathbb{H}_N(a) = \sum_{n=1}^{N} \frac{1}{F_{-n-a}},$$

$$\mathbb{I}_N(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n-a}}.$$

We write the above in terms of $\mathbb{F}_N(a)$ and $\mathbb{G}_N(a)$.

NON-ALTERNATING SUM

The non-alternating sum of Fibonacci numbers of negative indices of 1st order is,

$$\mathbb{H}_N(a) = \sum_{n=1}^{N} \frac{1}{F_{-n-a}}.$$

We write this sum in terms of $\mathbb{G}_N(a)$.

$$\mathbb{H}_N(a) = \sum_{n=1}^{N} \frac{1}{F_{-n-a}} = \sum_{n=1}^{N} \frac{1}{(-1)^{n+a+1}F_n} = \sum_{n=1}^{N} \frac{(-1)^{n+a+1}}{F_n} = (-1)^{a+1} \sum_{n=1}^{N} \frac{(-1)^n}{F_n}$$

$$= (-1)^{a+1}\mathbb{G}_N(a).$$

■

ALTERNATING SUM

The alternating sum of Fibonacci numbers of negative indices of 1st order is,

$$\mathbb{I}_N(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n-a}}.$$

We express this in terms of $\mathbb{F}_N(a)$.

$$\mathbb{I}_N(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n-a}} = \sum_{n=1}^{N} \frac{(-1)^n}{(-1)^{n+a+1}F_n} = \sum_{n=1}^{N} \frac{(-1)^{a+1}}{F_n} = (-1)^{a+1} \sum_{n=1}^{N} \frac{1}{F_n} = (-1)^{a+1}\mathbb{F}_N.$$

■

ORDER 2

We next consider the following 2nd order sums:

$$\mathbb{H}_N(0,a) = \sum_{n=1}^{N} \frac{1}{F_{-n}F_{-n-a}},$$

$$\mathbb{I}_N(0,a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n}F_{-n-a}},$$

$$\mathbb{H}_N(a,b) = \sum_{n=1}^{N} \frac{1}{F_{-n-a}F_{-n-b}},$$

$$\mathbb{I}_N(a,b) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n-a}F_{-n-b}}.$$

We express the above sums in terms of $\mathbb{F}_N(0,a)$, $\mathbb{G}_N(0,a)$, $\mathbb{F}_N(a,b)$ and $\mathbb{G}_N(a,b)$.

NON-ALTERNATING SUM

We consider the following 2nd order non-alternating sums.

$$\mathbb{H}_N(0,a) = \sum_{n=1}^{N} \frac{1}{F_{-n}F_{-n-a}} \; where \; a > 0,$$

$$\mathbb{H}_N(a,b) = \sum_{n=1}^{N} \frac{1}{F_{-n-a}F_{-n-b}} \; where \; 0 < a < b.$$

We express the above sums in terms of $\mathbb{F}_N(0,a)$ and $\mathbb{F}_N(a,b)$ respectively.

First we consider the sum,

$$\mathbb{H}_N(a) = \sum_{n=1}^{N} \frac{1}{F_{-n}F_{-n-a}} = \sum_{n=1}^{N} \frac{1}{(-1)^{2n+a+2}F_n} = \sum_{n=1}^{N} \frac{(-1)^a}{F_nF_{n+a}} = (-1)^a \sum_{n=1}^{N} \frac{1}{F_nF_{n+a}} = (-1)^a \mathbb{F}_N(0,a).$$

■

We next consider,

$$\begin{aligned}
\mathbb{H}_N(a,b) &= \sum_{n=1}^{N} \frac{1}{F_{-n-a}F_{-n-b}} \\
&= \sum_{n=1}^{N} \frac{1}{(-1)^{2n+a+b+2}F_{n+a}F_{n+b}} \\
&= \sum_{n=1}^{N} \frac{(-1)^{a+b}}{F_{n+a}F_{n+a}} \\
&= (-1)^{a+b} \sum_{n=1}^{N} \frac{1}{F_{n+a}F_{n+b}} \\
&= (-1)^{a+b}\mathbb{F}_N(a,b).
\end{aligned}$$

■

ALTERNATING SUM

We consider the following 2nd order alternating sums.

$$\mathbb{I}_N(0,a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n}F_{-n-a}} \; where \; a > 0,$$

$$\mathbb{I}_N(a,b) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-n-a}F_{-n-b}} \; where \; 0 < a < b.$$

We express the above sums in terms of $\mathbb{G}_N(0,a)$ and $\mathbb{G}_N(a,b)$.

First we consider the sum,

$$\mathbb{I}_N(a)=\sum_{n=1}^{N}\frac{(-1)^n}{F_{-n}F_{-n-a}}=\sum_{n=1}^{N}\frac{(-1)^n}{(-1)^{2n+a+2}F_n}=\sum_{n=1}^{N}\frac{(-1)^{n+a}}{F_nF_{n+a}}=(-1)^a\sum_{n=1}^{N}\frac{(-1)^n}{F_nF_{n+a}}=(-1)^a\mathbb{G}_N(0,a).$$

■

Next we consider,

$$\begin{aligned}\mathbb{I}_N(a,b)&=\sum_{n=1}^{N}\frac{(-1)^n}{F_{-n-a}F_{-n-b}}\\&=\sum_{n=1}^{N}\frac{(-1)^n}{(-1)^{2n+a+b+2}F_{n+a}F_{n+b}}\\&=\sum_{n=1}^{N}\frac{(-1)^{n+a+b}}{F_{n+a}F_{n+a}}\\&=(-1)^{a+b}\sum_{n=1}^{N}\frac{(-1)^n}{F_{n+a}F_{n+b}}\\&=(-1)^{a+b}\mathbb{G}_N(a,b).\end{aligned}$$

■

SUM WITH INDICES IN A.P.

Let h be a natural number. We convert the following sums in terms of $\mathbb{F}_N^h$ and $\mathbb{G}_N^h$.

$$\mathbb{H}_N^h=\sum_{n=1}^{N}\frac{1}{F_{-hn}}\,,\mathbb{I}_N^h=\sum_{n=1}^{N}\frac{(-1)^n}{F_{-hn}}.$$

$$\begin{aligned}\mathbb{H}_N^h&=\sum_{n=1}^{N}\frac{1}{F_{-hn}}\\&=\sum_{n=1}^{N}\frac{1}{(-1)^{hn+1}F_{hn}}\\&=\sum_{n=1}^{N}\frac{(-1)^{hn+1}}{F_{hn}}\\&=-\sum_{n=1}^{N}\frac{((-1)^h)^n}{F_{hn}}\end{aligned}$$

$$= \begin{cases} -\sum_{n=1}^{N} \frac{(-1)^n}{F_{hn}} & \text{if } h \text{ is odd,} \\ -\sum_{n=1}^{N} \frac{1}{F_{hn}} & \text{if } h \text{ is even.} \end{cases}$$

$$= \begin{cases} -\mathbb{G}_N^h & \text{if } h \text{ is odd,} \\ -\mathbb{F}_N^h & \text{if } h \text{ is even.} \end{cases}$$

■

$$\mathbb{I}_N^h = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-hn}}$$

$$= \sum_{n=1}^{N} \frac{(-1)^n}{(-1)^{hn+1} F_{hn}}$$

$$= \sum_{n=1}^{N} \frac{(-1)^{hn+1+n}}{F_{hn}}$$

$$= -\sum_{n=1}^{N} \frac{((-1)^{h+1})^n}{F_{hn}}$$

$$= \begin{cases} -\sum_{n=1}^{N} \frac{1}{F_{hn}} & \text{if } h \text{ is odd,} \\ -\sum_{n=1}^{N} \frac{(-1)^n}{F_{hn}} & \text{if } h \text{ is even.} \end{cases}$$

$$= \begin{cases} -\mathbb{F}_N^h & \text{if } h \text{ is odd,} \\ -\mathbb{G}_N^h & \text{if } h \text{ is even.} \end{cases}$$

■

NON-ALTERNATING SUM OF ORDER 1

We consider the problem of finding 1st order non-alternating sum of Fibonacci number with negative indices, $\mathbb{H}_N^h(a)$.

$$\mathbb{H}_N^h(a) = \sum_{n=1}^{N} \frac{1}{F_{-hn-a}} \text{ where } h > 0 \text{ and } h|a.$$

We write the above sum in terms of $\mathbb{F}_N(a)$ and $G_N(a)$.

$$
\begin{aligned}
\mathbb{H}_N^h(a) &= \sum_{n=1}^{N} \frac{1}{F_{-hn-a}} \\
&= \sum_{n=1}^{N} \frac{1}{(-1)^{hn+a+1} F_{hn+a}} \\
&= \sum_{n=1}^{N} \frac{(-1)^{hn+a+1}}{F_{hn+a}} \\
&= (-1)^{a+1} \sum_{n=1}^{N} \frac{(-1)^{hn}}{F_{hn+a}} \\
&= (-1)^{a+1} \sum_{n=1}^{N} \frac{((-1)^{h})^{n}}{F_{hn+a}} \\
&= (-1)^{a+1} \begin{cases} \displaystyle\sum_{n=1}^{N} \frac{(-1)^{n}}{F_{hn+a}} & \text{if } h \text{ is odd,} \\ \displaystyle\sum_{n=1}^{N} \frac{1}{F_{hn+a}} & \text{if } h \text{ is even.} \end{cases} \\
&= \begin{cases} (-1)^{a+1} \mathbb{G}_N^h(a) & \text{if } h \text{ is odd,} \\ (-1)^{a+1} \mathbb{F}_N^h(a) & \text{if } h \text{ is even.} \end{cases}
\end{aligned}
$$

■

ALTERNATING SUM OF ORDER 1

We consider the problem of finding 1st order non-alternating sum of Fibonacci number with negative indices, $\mathbb{I}_N^h(a)$.

$$\mathbb{I}_N^h(a) = \sum_{n=1}^{N} \frac{(-1)^n}{F_{-hn-a}} \text{ where } h > 0 \text{ and } h|a.$$

We express the above sum in terms of $\mathbb{F}_N(a)$ and $G_N(a)$.

$$
\begin{aligned}
\mathbb{I}_N^h(a) &= \sum_{n=1}^{N} \frac{(-1)^n}{F_{-hn-a}} \\
&= \sum_{n=1}^{N} \frac{(-1)^n}{(-1)^{hn+a+1} F_{hn+a}} \\
&= \sum_{n=1}^{N} \frac{(-1)^{hn+a+1}}{F_{hn+a}}
\end{aligned}
$$

$$= (-1)^{a+1}\sum_{n=1}^{N}\frac{(-1)^{hn+n}}{F_{hn+a}}$$

$$= (-1)^{a+1}\sum_{n=1}^{N}\frac{\left((-1)^{(h+1)}\right)^{n}}{F_{hn+a}}$$

$$= (-1)^{a+1}\begin{cases}\displaystyle\sum_{n=1}^{N}\frac{1}{F_{hn+a}} & \text{if } h \text{ is odd,}\\ \displaystyle\sum_{n=1}^{N}\frac{(-1)^{n}}{F_{hn+a}} & \text{if } h \text{ is even.}\end{cases}$$

$$= \begin{cases}(-1)^{a+1}\mathbb{F}_N^h(a) & \text{if } h \text{ is odd,}\\ (-1)^{a+1}\mathbb{G}_N^h(a) & \text{if } h \text{ is even.}\end{cases}$$

■

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